Third review for the Final Exam
Mat 323, Fall 2005

1. Find the Laplace transforms of

(a) $f(t) = 2t^3 e^{2t}$

(b)
$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $e^t\, u(t-1)$

(d) $e^{-t}\, t \sin(2\,t)$

(a) $\frac{12}{(s-2)^4}$

(b) $f(t) = t^2 - \left((t-1)^2 + 2(t-1) + 1\right) u(t-1)$

$F(s) = \frac{2}{s^3} - \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) e^{-s}$

(c) $\frac{e^{-s-1}}{s-1}$

(d) $\mathcal{L}\left(e^{-t}\sin(2t)\right) = \frac{2}{(s-1)^2+4}$

$\mathcal{L}\left(te^{-t}\sin(2t)\right) = -\left(\frac{2}{(s-1)^2+4}\right)' = \frac{4(s-1)}{((s-1)^2+4)^2}$

2. Find the inverse Laplace transforms of Find $f(t)$ for

(a) $F(s) = \dfrac{(3\,s+5)\,e^{-s}}{s\,(s^2+s-6)}$

(b) $F(s) = \ln(s-1)$ (hint: calculate the inverse Laplace transform of $F'(s)$ and use the last formula in the tables).

(a) $$\frac{3s+5}{s(s-2)(s+3)} = \frac{11}{10(s-2)} - \frac{4}{15(s+3)} - \frac{5}{6s}$$

$$f(t) = u(t-1)\left(\frac{11}{10}e^{2(t-1)} - \frac{4}{15}e^{-3(t-2)} - \frac{5}{6}\right)$$

(b) $$F' = \frac{1}{s-1}$$

$$\left.\begin{aligned} \mathcal{L}^{-1}(F') &= e^{t} \\ \mathcal{L}^{-1}(F') &= -t\,f(t) \end{aligned}\right\} \quad f(t) = -\frac{e^{t}}{t}$$

3. Solve the following differential equation

$$y'' + 4y' + 6\,y = g(t), \quad y(0) = 0, \quad y'(0) = 1$$

where

$$g(t) = \begin{cases} 1 & \text{for } 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$y'' + 4y' + 6y = u(t-1) - u(t-2)$$

$$(s^2 + 4s + 6)\,Y = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$Y = \frac{e^{-s}}{s(s^2+4s+6)} - \frac{e^{-2s}}{s(s^2+4s+6)}$$

Here, I had the wrong numbers in class sorry ☹

$$\frac{1}{s(s^2+4s+6)} = \frac{1}{6s} - \frac{s+4}{6(s^2+4s+6)}$$

$$= \frac{1}{6s} - \frac{s+4}{6((s+2)^2+2)}$$

$$= \frac{1}{6s} - \frac{(s+2)+2}{6((s+2)^2+2)}$$

$$= \frac{1}{6s} - \frac{1}{6}\,\frac{s+2}{(s+2)^2+2} - \frac{1}{3\sqrt{2}}\,\frac{\sqrt{2}}{(s+2)^2+2}$$

$$\rightarrow \mathcal{L}^{-1}\left(\frac{1}{s(s^2+4s+6)}\right) = \frac{1}{6} - \frac{e^{-2t}}{6}\cos(\sqrt{2}t) - \frac{e^{-2t}}{3\sqrt{2}}\sin(\sqrt{2}t)$$

$$\rightarrow y = u(t-1)\left[\frac{1}{6} - \frac{e^{-2(t-1)}}{6}\cos(\sqrt{2}(t-1)) - \frac{e^{-2(t-1)}}{3\sqrt{2}}\sin(\sqrt{2}(t-1))\right]$$

$$- u(t-2)\left[\frac{1}{6} - \frac{e^{-2(t-2)}}{6}\cos(\sqrt{2}(t-2)) - \frac{e^{-2(t-2)}}{3\sqrt{2}}\sin(\sqrt{2}(t-2))\right]$$

4. Find the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the initial value problem

$$(2x - 3)y'' - x\,y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

Note: trying to do the recurrence formula here would be difficult!

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$(2x-3)\left(2a_2 + 6a_3 x + \ldots\right)$$

$$-x\left(a_1 + 2a_2 x + 3a_3 x^2 + \ldots\right)$$

$$+\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right)$$

$$= \left(-6a_2 + a_0\right) + x\left(4a_2 - 18a_3 - a_1 + a_1\right) + \ldots$$

do not need more terms to have what we need

$$a_2 = \frac{a_0}{6} \qquad \frac{4a_0}{6} - 18a_3 = 0$$

$$a_3 = \frac{2a_0}{54} = \frac{a_0}{27}$$

$$y(0) = 1,\ y'(0) = 2 \longrightarrow a_0 = 1,\ a_1 = 2,\ a_2 = \frac{1}{6},\ a_3 = \frac{1}{27}$$

$$\boxed{y = 1 + 2x + \frac{1}{6}x^2 + \frac{1}{27}x^3 + \ldots}$$

5. Find the recurrence relation for the power series solution (around $x = 0$) of

$$2y'' + xy' + 3y = 0.$$

Then use the recurrence to find the first 6 terms in the series for y.

$$y = \sum_{n=0}^{\infty} a_n x^n \to \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$p = n-2$

$$\sum_{p=0}^{\infty} 2(p+2)(p+1)a_{p+2}x^p + \sum_{r=1}^{\infty} r a_r x^r + \sum_{q=1}^{\infty} 3a_q x^q + 3a_0 = 0$$

$$\sum_{s=1}^{\infty} 2(s+2)(s+1)a_{s+2}x^s + \sum_{s=1}^{\infty} s a_s x^s + \sum_{s=1}^{\infty} 3a_s x^s + 3a_0 + 4a_2 = 0$$

$$\sum_{s=1}^{\infty} \left[2(s+2)(s+1)a_{s+2} + (s+3)a_s\right]x^s + 3a_0 + 4a_2 = 0$$

$$\boxed{a_{s+2} = \frac{-(s+3)a_s}{2(s+2)(s+1)} \quad s \geq 1 \quad a_2 = -\frac{3}{4}a_0}$$

$a_2 = -\frac{3}{4}a_0$, $a_3 = \frac{-4a_1}{12} = -\frac{a_1}{3}$, $a_4 = \frac{-5a_2}{24} = \frac{15a_0}{96} = \frac{5a_0}{32}$

$a_5 = -\frac{6a_3}{40} = \frac{24a_1}{480} = \frac{a_1}{20}$

$$\boxed{y = a_0 + a_1 x - \frac{3}{4}a_0 x^2 - \frac{a_1}{3}x^3 + \frac{5a_0}{32}x^4 + \frac{a_1}{20}x^5 + \ldots}$$

System of equations: p. 541, problems 12,14,34; p. 549, problem 2, 14