Say whether the following ode's are linear or nonlinear (L or NL), exact or not (E or NE), separable or not (S or NS).

	Equation	L or NL	E or NE	S or NS
1	$t^2\frac{dy}{dt} + ty = t$	L	NE	S
2	$3t = e^t\frac{dy}{dt} + y\ln t$	L	NE	NS
3	$y^2\,dx = -(2xy + \cos y)\,dy$	NL	E	NS
4	$y\,dx = x\,dy$	L	NE	S
5	$3r = \frac{dr}{d\theta} - \theta^3$	NL	NE	S
6	$\frac{dy}{dx} = \frac{3x^2+4x+2}{2y+1}$	NL	(*)	S
7	$\frac{d^2y}{dt^2} + ty\frac{dy}{dt} = 0$	NL	N/a.	N/a
8	$\frac{dy}{dx} = 5$	L	E	S

Solve the initial value problem

$$\frac{dy}{dx} - 2xy = 0 \qquad y(0) = 1$$

$\frac{dy}{y} = 2x\,dx$ $\quad \ln|y| = x^2 + c$

$y = ke^{x^2}$ $\quad y(0) = 1 \to k = 1$

$y(x) = \boxed{e^{x^2}}$

Hint. You will find $y(\sqrt{\ln 2}) = 2$

(*) Written as $(2y+1)dy - (3x^2+4x+2)dx = 0$ it is exact

Solve the following differential equations

$$(x+1)\,dy = (\cos x - y)\,dx$$

$(x+1)dy + (y-\cos x)\,dy = 0 \rightarrow$ Exact

$\frac{\partial F}{\partial y} = x+1$ (1) $\qquad$ (1) $\rightarrow F = (x+1)y + g(x)$

Plug into (2) $\rightarrow y + g' = y - \cos x$

$\frac{\partial F}{\partial x} = y - \cos x$ (2) $\qquad g' = -\cos x \rightarrow g = \sin x$

$\rightarrow \boxed{(x+1)y - \sin x = C}$

$$x\frac{dv}{dx} = \frac{1-4v^2}{3v}$$

Separation: $\frac{dx}{x} = \frac{3V}{1-4V^2}\,dV$

$\ln|x| = -\frac{3}{8}\ln|1-4V^2| \rightarrow 1-4V^2 = C_1 X^{-8/3}$

$\rightarrow V^2 = C_2 X^{-8/3} + \frac{1}{4}$

$\boxed{V = \sqrt{C_2 X^{-8/3} + \frac{1}{4}}}$

$$y' = x^3(1-y), \quad y(0) = 3$$

Separation: $\frac{dy}{1-y} = x^3\,dx$

$-\ln|1-y| = \frac{x^4}{4} + C$

$1-y = Ke^{-\frac{x^4}{4}}$

$y(0) = 3 \rightarrow K = -2 \qquad \rightarrow \boxed{y = 2e^{-x^4/4} + 1}$

$$\frac{dy}{dx} = x - 2xy$$

linear $\frac{dy}{dx} + 2xy = x$ integating factor $\mu = e^{\int 2x dx} = e^{x^2}$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2} y\right) = x e^{x^2}$$

Integrate: $e^{x^2} y = \frac{e^{x^2}}{2} + C \rightarrow \boxed{y = \frac{1}{2} + Ce^{-x^2}}$

$$\cos x \frac{dy}{dx} = -y \sin x + 2x \cos^2 x$$

$\frac{dy}{dx} + y \frac{\sin(x)}{\cos(x)} = 2x\cos x \leftarrow$ standard form

Integrating factor $\exp\left(\int \frac{\sin(x)}{\cos(x)} dx\right) = \exp(-\ln(\cos(x)))$

$$= \frac{1}{\cos(x)}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{\cos x}\right) = 2x$$

$\frac{y}{\cos(x)} = x^2 + C$ $\boxed{y = x^2 \cos x + C\cos x}$

$$(1/y)\,dx + (3y - x/y^2)\,dy = 0$$

Exact $\left.\begin{array}{l} \frac{\partial F}{\partial x} = \frac{1}{y} \\ \frac{\partial F}{\partial y} = 3y - \frac{x}{y^2} \end{array}\right\} F = \frac{3y^2}{2} + \frac{x}{y} + C$

$$\boxed{\frac{3y^2}{2} + \frac{x}{y} = C}$$

The population $P(t)$, measured in units of 1000, of a species obeys the ode

$$\frac{dP}{dt} = 2P(1-P)$$

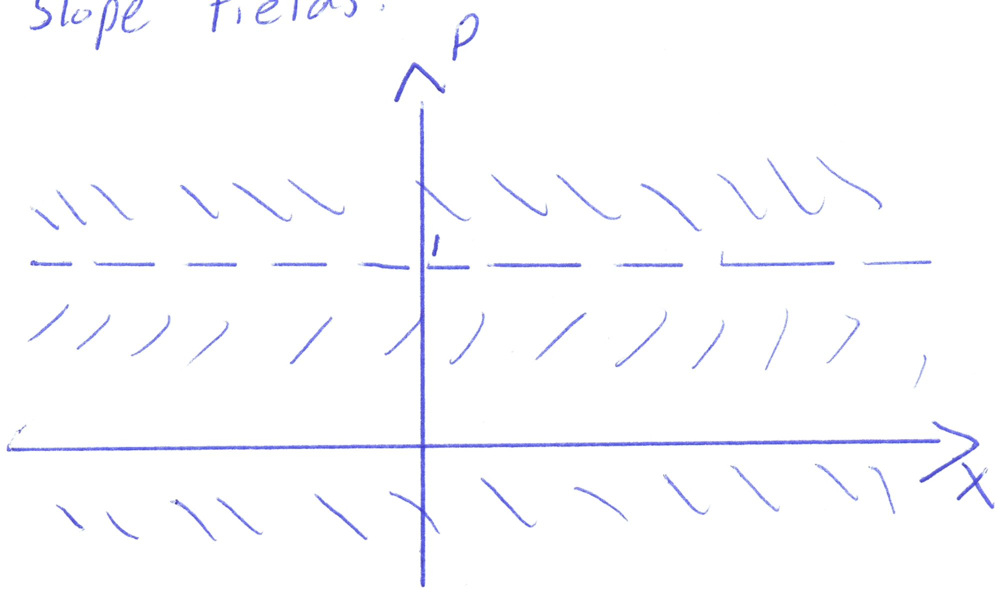
where time t is measured in years.

If the starting population was 2000, i.e. $P(0) = 2$,

To which value does $P(t)$ tend as $t \to \infty$?

$$\lim_{t\to\infty} P(t) = 1$$

Slope fields:



The amount of radioactive isotope $Q(t)$ (measured in kgs. or as a percentage of its original value) obeys the ode

$$\frac{dQ(t)}{dt} = -kQ(t)$$

where t, time is measured in years. Given $Q(0) = Q_0$, find $Q(t)$ in terms of Q_0, k and t.

$$\boxed{Q(t) = Q_0 e^{-kt}}$$

The half-life τ is the time it takes the amount $Q(t)$ to decay to $\frac{1}{2}Q_0$; i.e. $Q(\tau) = \frac{1}{2}Q_0$. Find an expression for τ in terms of k, the decay rate

$$\boxed{\tau = -\frac{1}{k}\ln\left(\frac{1}{2}\right)}$$

$$Q_0 e^{-k\tau} = \frac{1}{2}Q_0 \to e^{-k\tau} = \frac{1}{2}$$

$$\to \tau = -\frac{1}{k}\ln\left(\frac{1}{2}\right)$$

A bone is found on an archeological dig. When found, the amount of radioactive isotope C^{14} (carbon 14) was 25% of what it was when the bone was part of a living organism. Given the half life of C^{14} is 5600 years, how long has the bone been dead (age)?

$$\boxed{Age = 11200}$$

Find k: $\tau = -\frac{1}{k}\ln\left(\frac{1}{2}\right)$ $\quad \tau = 5600$ yrs

$$k = -\frac{1}{5600}\ln\left(\frac{1}{2}\right)$$

Let T be the age

$$Q_0 e^{-kT} = 0.25\, Q_0$$

$$T = -\frac{1}{k}\ln(0.25) = 11200$$

Consider the ode

$$\frac{dy}{dt} = y(1-y)(2-y)$$

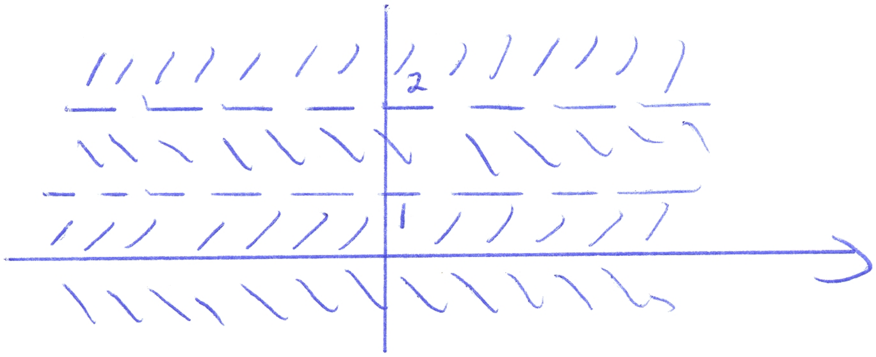
Given (a) $0 < y(0) < 1$, $y(t) \to$ $\boxed{1}$ as $t \to \infty$.

Given (b) $1 < y(0) < 2$, $y(t) \to$ $\boxed{1}$ as $t \to \infty$.

Given (c) $y(0) > 2$, $y(t) \to$ $\boxed{\infty}$ as $t \to \infty$.

Which of the equilibrium points $y = 0$, $y = 1$, and $y = 2$ is stable?

y=1 only



Model for an object falling toward earth. Assuming that only air and gravity are acting on the object, the velocity v is given by

$$m\frac{dv}{dt} = m\,g - bv$$

where m is the mass of the object, g is the acceleration due to gravity and $b > 0$ is constant. If $m = 100kg$, $g = -9.8m/sec^2$, $b = 5kg/sec$, $v(0) = 10m/sec$, solve for $v(t)$. What is the terminal velocity?

Solve it using the fact it is linear

$$\frac{dv}{dt} + \frac{b}{m}v = g \qquad \text{standard form}$$

Integrating factor $e^{\frac{b}{m}t}$

$$\frac{d}{dt}\left(e^{\frac{b}{m}t}v\right) = g e^{\frac{b}{m}t}$$

$$e^{\frac{b}{m}t}v = \frac{gm}{b}e^{\frac{b}{m}t} + C$$

$$V = \frac{gm}{b} + Ce^{-\frac{b}{m}t}$$

$g=-9.8,\ b=5,\ m=100 \rightarrow V = -196 + Ce^{-0.05t}$

$V(0)=10 \rightarrow C=206$ $\qquad \boxed{V = -196 + 206e^{-0.05t}}$

Terminal velocity corresponds to the limit as $t \to \infty$.

$\rightarrow$ Stationary solutions: $\quad mg - bv = 0$

$$V = -\frac{980}{5} = \boxed{-196\ m/s}$$

Newton's law of cooling:

$$\frac{dH}{dt} = k(T - H)$$

where H is the temperature of the object, $k > 0$ is constant, and T is the temperature of the surrounding.

Solve for H.

A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 minutes, the thermometer reads 80°F. What is the reading after 20 minutes?

$$H = ce^{-kt} + T$$

$$T = 70, \quad H(0) = 100$$

$$H = 30e^{-kt} + 70$$

$$H(6) = 80 = 30e^{-k6} + 70$$

$$k = -\frac{1}{6}\ln\left(\frac{1}{3}\right)$$

$$H(20) = 30e^{\frac{20}{6}\ln\left(\frac{1}{3}\right)} + 70 \approx 70.77$$

Consider the following non-homogeneous equations. What is the form of a particular solutions? (**you do not need to find the value of the coefficients**). The particular solution found using the undetermined coefficient method.

a) $y'' + 25\, y = \cos(3\, t)$ $r = \pm i$

$3i$ is not a root

$y_p = A\cos(3t) + B\sin(3t)$

b) $2\, y'' + 128\, y = \sin(8\, t)$ $r = \pm 8i \rightarrow 8i$ is a root

$y_p = tA\sin(8t) + tB\cos(8t)$

c) $y'' + 2\, y' - 3\, y = 2\, te^t \sin t$ $r_1 = -3,\ r_2 = 1$

$1+i$ is not a root

$y_p = (At+B)e^t\sin t + (Ct+D)e^t\cos t$

d) $y'' + 2\, y' - 3\, y = 5\, t^2\, e^{-3\, t}$ $r_1 = -3,\ r_2 = 1$

-3 is a root

$y_p = t(At^2+Bt+C)e^{-3t}$

e) $y'' - 4\, y' + 13\, y = \cos(3\, t) + e^{2t}$ $r = 2 \pm 3i$

2 and $3i$ are not roots $\rightarrow y_p = A\cos(3t) + B\sin(3t) + Ce^{2t}$

f) $y'' - 4\, y' + 13\, y = t^2\, e^{2\, t}\cos(3\, t) + \sin(3\, t)$ $r = 2 \pm 3i$

$2+3i$ is a root

$3i$ is not

$y_p = t(At^2+Bt+C)e^{2t}\cos(3t) + t(Dt^2+Et+F)e^{2t}\sin(3t)$

$+ G\sin(3t) + H\cos(3t)$

g) $y'' - 6\, y' + 9\, y = t\, e^{3\, t}$

$r_1 = r_2 = 3$

3 is a double root $y_p = t^2(At+B)e^{3t}$

A particular solution to

$$y'' - 4y' + 13y = e^t$$

is given by $y_p = \frac{1}{10} e^t$.

a) Find the general solution to the equation.

b) Find the solution that satisfies the conditions $y(0) = 1$ and $y'(0) = 0$.

(a) $r = 2 \pm 3i$

$$y_g = \frac{1}{10}e^t + C_1 e^{2t} \cos(3t) + C_2 e^{2t} \sin(3t)$$

(b) $y(0) = 1 \rightarrow \frac{1}{10} + C_1 = 1 \qquad C_1 = \frac{9}{10}$

$$y' = \frac{1}{10}e^t + 2C_1 e^{2t}\cos(3t) - 3C_1 e^{2t} \sin(3t) + 2C_2 e^{2t} \sin(3t) + 3C_2 e^{2t} \cos(3t)$$

$y'(0) = 0$

$\rightarrow \frac{1}{10} + 2C_1 + 3C_2 = 0$

$C_2 = -\frac{19}{30}$

$$\boxed{y = \frac{1}{10}e^t + \frac{9}{10}e^{2t}\cos(3t) - \frac{19}{30}e^{2t}\sin(3t)}$$