REVIEW FOR EXAM #2 MATH 323

Note: to make sure you know all the cases, review #15, 16 on p. 331

1) Find the general solution of the equations

a) $y^{(4)} - 7\,y''' + 17\,y'' - 17\,y' + 6y = 0$ (e^x and $x\,e^x$ are solutions).

Long division by $(r-1)^2$ because e^x and xe^x are solutions

$$r^4 - 7r^3 + 17r^2 - 17r + 6 = (r-1)^2(r-2)(r-3)$$

$$y = c_1 x e^x + c_2 e^x + c_3 e^{2x} + c_4 e^{3x}$$

b) $y^{(4)} - 4\,y''' + 7\,y'' - 16\,y' + 12\,y = 0$ ($\sin(2\,x)$ is a solution).

r^2+4 is a factor because $\sin(2x)$ is a solution

$$r^4 - 4r^3 + 7r^2 - 16r + 12 = (r-1)(r-3)(r^2+4)$$

$$y = c_1 e^x + c_2 e^{3x} + c_3 \sin(2x) + c_4 \cos(2x)$$

c) $y''' - 6\,y'' + 21\,y' - 26y = 0$ ($e^{2x}\sin(3\,x)$ is a solution).

$(r-2-3i)(r-2+3i) = (r^2-4r-13)$ is a factor

$$r^3 - 6r^2 + 21r - 26 = (r-2)(r^2-4r+13)$$

$$y = c_1 e^{2x} + c_2 e^{2x}\sin(3x) + c_3 e^{2x}\cos(3x)$$

2) Use the reduction of order to find a second solution

$$t^2\,y'' + 2\,t\,y' - 2\,y = 0, \;\; y_1 = t$$

$y = vt$

$y' = v't + v$

$y'' = v''t + 2v'$

$\rightarrow t^2(v''t + 2v') + 2t(v't + v) - 2vt = 0$

$v''t^3 + v' \cdot 4t^2 = 0$

$\frac{v''}{v'} = -\frac{4}{t}$

$\ln v' = -4 \ln t$

$v' = t^{-4}$

$v = -\frac{1}{3t^3}$

$y_2 = \frac{t}{t^3} = \frac{1}{t^2}$

3) The differential equation

$$x^2y'' + xy' - \left(\frac{1}{4} + x^2\right) y = 0$$

has linearly independent solutions $y_1(x) = \dfrac{e^{-x}}{\sqrt{x}}$ and $y_2(x) = \dfrac{e^x}{\sqrt{x}}$ for $x > 0$.

(1) Find the Wronskian of the two solutions above.

(2) Use the method of variation of parameters to find a particular solution of

$$x^2y'' + xy' - \left(\frac{1}{4} + x^2\right) y = x^{5/2},$$

on $(0, \infty)$. (Use the formulae seen in Section 6.4 to calculate v_1' and v_2'. To calculate W_1 for example, take the Wronskian and replace the first column by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Remember the standard form before using the formulae.)

Hint to make sure you are on the right track: the Wronskian is $2/x$.

(1) $$W = \begin{vmatrix} \frac{e^{-x}}{\sqrt{x}} & \frac{e^{x}}{\sqrt{x}} \\ -\frac{e^{-x}}{\sqrt{x}} - \frac{e^{-x}}{2x^{3/2}} & \frac{e^{x}}{\sqrt{x}} - \frac{e^{x}}{2x^{3/2}} \end{vmatrix}$$

$$= \frac{e^{-x}}{\sqrt{x}}\left(\frac{e^{x}}{\sqrt{x}} - \cancel{\frac{e^{x}}{2x^{3/2}}}\right) + \frac{e^{x}}{\sqrt{x}}\left(\frac{e^{-x}}{\sqrt{x}} + \cancel{\frac{e^{-x}}{2x^{3/2}}}\right)$$

$$= \frac{2}{x}$$

(2) $$W_1 = \begin{vmatrix} 0 & \frac{e^{x}}{\sqrt{x}} \\ 1 & \frac{e^{x}}{\sqrt{x}} - \frac{e^{x}}{2x^{3/2}} \end{vmatrix} = -\frac{e^{x}}{\sqrt{x}}$$

$$W_2 = \begin{vmatrix} \frac{e^{-x}}{\sqrt{x}} & 0 \\ -\frac{e^{-x}}{\sqrt{x}} - \frac{e^{-x}}{x^{3/2}} & 1 \end{vmatrix} = \frac{e^{-x}}{\sqrt{x}}$$

$$Y = V_1 Y_1 + V_2 Y_2$$

$$g(x) = \frac{x^{5/2}}{x^2} = x^{1/2}$$

$$\rightarrow V_1' = \frac{x^{1/2}\left(-\frac{e^x}{\sqrt{x}}\right)}{2/x} = \frac{-xe^x}{2}$$

$$V_1 = -\frac{1}{2}\int xe^x\,dx = -\frac{1}{2}\left[xe^x - \int e^x\,dx\right]$$

$$V_1 = -\frac{1}{2}(x-1)e^x$$

$$V_2' = \frac{g\,W_2}{W} = \frac{xe^{-x}}{2}$$

$$V_2 = \frac{1}{2}\int xe^{-x}\,dx = \frac{1}{2}\left[-xe^{-x} + \int e^{-x}\,dx\right]$$

$$= -\frac{1}{2}(x+1)\,e^{-x}$$

$$y_p = V_1 Y_1 + V_2 Y_2 = -\frac{1}{2\sqrt{x}}\left(x-1+x+1\right) = \frac{-2x}{2\sqrt{x}} = -\sqrt{x}$$

$$\boxed{y_p = -\sqrt{x}}$$

4) Find the values of α such that the functions $y_1 = x^2 - \alpha x + 1$, $y_2 = x^2 + x$, and $y_3 = 1$ are linearly dependent.

$$W = \begin{vmatrix} x^2 - \alpha x + 1 & x^2 + x & 1 \\ 2x - \alpha & 2x + 1 & 0 \\ 2 & 2 & 0 \end{vmatrix}$$

$$= -2\alpha - 2$$

$$W = 0 \Longleftrightarrow -2\alpha - 2 = 0$$

$$\boxed{\alpha = -1}$$

5. (12 pts) *Allow me to introduce myself...my name is Wile. E. Coyote...genius*
After many unsucessfull and painful attempts Wile E. Coyote (20 kg) has nally found the ultimate plan to catch MeepMeep the Road Runner. The main device in his ingenious scheme consists of a compressed horizontal giant spring. Wile E. Coyote will wear roller skates and will be pushed by the compressed spring and so doing gather enough speed to catch MeepMeep (Coyote: Brilliance, thats all I can say. Sheer Unadulterated Brilliance!!!).

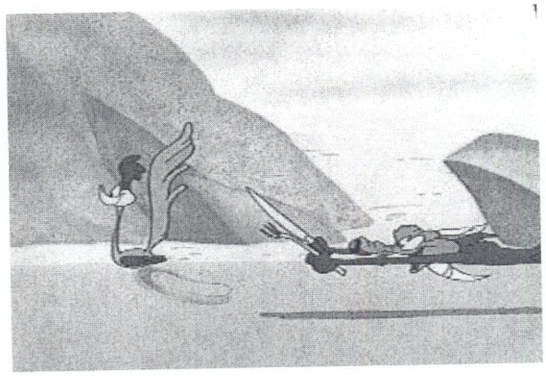

FIGURE 1. In action...

According to the manufacturer, ACME, the giant spring has a Hookean constant of $k = 160$ (Newtons m^{-1}) and a damping coefficient of $b = 80$ (Newtons s m^{-1}). Do not forget the mass of the coyote, $m = 20$ kg. The spring is initially compressed by Wile 5 meters from its equilibrium position ($x(0) = -5, \;\; x'(0) = 0$). At time t = 0, MeepMeep is seen by Coyote and he releases the spring...

(a) Let x(t) be the displacement of the spring from rest at time t. Write down the differential equation for the motion. Say if it is an underdamped or overdamped system.

Equation: $20x'' + 80x' + 160x = 0$

Underdamped or overdamped? $80^2 - 4 \cdot 20 \cdot 160 < 0$
$= -6400$

(b) Solve for $x(t)$ with the given initial conditions.

$$\boxed{x(t)= e^{-2t}\left(-5\cos(2t)-5\sin(2t)\right)}$$

$$x = e^{-\alpha t}\left[C_1\cos(\beta t)+C_2\sin(\beta t)\right]$$

$$\alpha=\frac{b}{2m}=2 \qquad \beta=\frac{\sqrt{4mk-b^2}}{2m}=2$$

$$x=e^{-2t}\left[C_1\cos(2t)+C_2\sin(2t)\right]$$

$$\left.\begin{array}{l} x(0)=-5 \\ x'(0)=-5 \end{array}\right\} \rightarrow C_1=C_2=-5$$

(c) Find the time when the speed is first maximal.

$$\boxed{t=\pi/8}$$

$$x'=20e^{-2t}\sin(2t)$$

$$x''=40e^{-2t}\left[\cos(2t)-\sin(2t)\right]$$

To maximise x' : $x''=0 \rightarrow \tan(2t)=1$

$$2t=\pi/4 \qquad t=\pi/8$$

(d) What is Wile E. Coyotes maximal velocity? Knowing that the velocity of MeepMeep is 6.5 m/s, will Wile E. Coyote catch his prey?

$$x'(\pi/8)=20e^{-\pi/4}\frac{\sqrt{2}}{2}\approx 6.45$$

No (of course!)

6) Consider the following oscillators. What is the form of the particular solutions? **(you do not need to find the value of the coefficients)**.

a) $y'' + 25\,y = \cos(3\,t)$

$y_p = A\cos(3t) + B\sin(3t)$

$3i$ is not a root of the characteristic polynomial

b) $2\,y'' + 128\,y = \sin(8\,t)$

$y_p = t(A\sin(8t) + B\cos(8t))$

$8i$ is a root of the characteristic polynomial

7) Consider the mass-spring system governed by

$$y'' + b\,y' + 16\,y = 0.$$

In both cases $b = 6$ and $b = 10$, say if the system is overdamped or underdamped and write down the solution satisfying $y(0) = 1$ and $y'(0) = 0$. In the underdamped case, write the solution in the *amplitude-phase* form.
Vocabulary: the *amplitude-phase* form means the form $A\,e^{\alpha t}\,\sin(\beta t + \phi)$.

(i) $b=6$

$b^2-4mk = \underset{-28}{\underset{\|}{36-64}} < 0$ underdamped

$$y = e^{-3t}\left[c_1\cos(\sqrt{7}t) + c_2\sin(\sqrt{7}t)\right]$$

$y(0)=1$, $y'(0)=0$ $\quad c_1 = 1$, $c_2 = \frac{3}{\sqrt{7}}$

$$y = e^{-3t}\left[\cos(\sqrt{7}t) + \frac{3}{\sqrt{7}}\sin(\sqrt{7}t)\right]$$

$$A = \sqrt{1+\frac{9}{7}} = \sqrt{\frac{16}{7}} = \frac{4}{\sqrt{7}}$$

$\tan(\varphi) = \frac{\sqrt{7}}{3}$

$\arctan\left(\frac{\sqrt{7}}{3}\right) \approx 0.723$

1st quadrant $\quad \varphi \approx 0.723$

$$\boxed{y = \frac{4}{\sqrt{7}}e^{-3t}\sin(\sqrt{7}t+\varphi)}$$

(ii) $b=10$ $\quad b^2-4mk = 36$ $\quad$ overdamped

$$y = c_1 e^{-2t} + c_2 e^{-8t}$$

$y(0)=1$, $y'(0)=0 \longrightarrow c_1 = \frac{4}{3}$, $c_2 = -\frac{1}{3}$

$$\boxed{y = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}}$$

8) Find the Laplace transforms of

$$f(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(s) = \int_0^1 t e^{-st}\, dt$$

$$= -\frac{t e^{-st}}{s} \Big|_0^1 + \frac{1}{s} \int_0^1 e^{-st}\, dt$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2} e^{-st} \Big|_0^1$$

$$F(s) = -\frac{e^{-s}}{s} - \frac{e^{-s} - 1}{s^2}$$

9) Find the inverse Laplace transforms of (you can use the table I distributed in class)

(a) $F(s) = \dfrac{3\,s+2}{2s^2+8s+10}$

(b) $F(s) = \dfrac{s-1}{(s-2)\,(s+1)}$

(c) $F(s) = \dfrac{s^2+9s+2}{(s-1)^2\,(s+3)}$

(d) $F(s) = \dfrac{s^2+10s}{(s+1)\,(s^2-2s+5)}$

(e) $F(s) = \ln(s-1)$ (hint: calculate the inverse Laplace transform of $F'(s)$ and use the last formula in your Table of Laplace transforms).

(a) $f(t) = \frac{3}{2}e^{-2t}\cos(t) - 2e^{-2t}\sin(t)$

(b) $f(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$

(c) $f(t) = (2+3t)e^{t} - e^{-3t}$

(d) $f(t) = -\frac{9}{8}e^{-t} + \frac{17}{8}e^{t}\cos(2t) + \frac{31}{8}e^{t}\sin(2t)$

(e) $F'(s) = \dfrac{1}{s-1}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s-1}\right) = e^{t}$

$\mathcal{L}^{-1}\left(F'(s)\right) = -t\,f(t)$

$\rightarrow -t\,f(t) = e^{t}$ $\quad \boxed{f(t) = -\dfrac{e^{t}}{t}}$

Not covered on the test

For fun then